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MULTIPLE SOLUTIONS FOR LAGRANGIAN SYSTEMS IN T^n

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ABSTRACT

In this paper we study existence of multiple periodic solutions of a general Lagrangean system having as a configuration space the n-dimensional torus T^n. The critical points of the potential energy correspond to equilibrium of the system when no external forces are present. We study the number of T-periodic solutions of the system, inherited by the equilibrium solutions, when the external force is not zero. In particular we prove that the forced system has at least the same number of periodic solutions as critical points the potential has, when certain condition is satisfies.

The proof of our results are based of the notion of Ljusternik-Schnirelmann relative category. We relate the level sets of the potential with the level sets of the functional associated to the Lagrangean system.

We also apply the ideas developed to study some multiplicity result for the existence of solutions of an elliptic partial differential equation with Neumann boundary condition.

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0 Introduction

Let us consider the ordinary differential equation

$$\ddot{q} + V_q(q) = f(t) \tag{0.1}$$

where $q = (q_1, \dots, q_n) \in \mathbf{R}^n$, and V satisfies

(V) $V \in C^2(\mathbf{R}^n, \mathbf{R})$ is τ_i-periodic in the q_i variable, $1 \leq i \leq n$.

When $f(t) \equiv 0$, (0.1) possesses at least $n+1$ equilibrium solutions, obtained as the critical points of the function V. This result is classical and is obtained regarding V as a function on the n-dimensional torus T^n, and using the fact that the Ljusternik-Schnirelmann category of T^n is $n+1$.

When f is not necessarily equal to 0 and it satisfies

(f) $f \in C(\mathbf{R}, \mathbf{R}^n)$ is T-periodic and

$$\frac{1}{T}\int_0^T f(t)\, dt = 0,$$

then it has been proved in [1], [5] and [11] that (0.1) possesses at least $n+1$ T-periodic solutions.

However the function V regarded as a function in T^n may have more that $n+1$ critical points. For example if V is a Morse function it is known that V possesses at least 2^n critical points. In this paper we give some conditions on V and f so that if V has ℓ critical points then (0.1) possesses at least ℓ T-periodic solutions. We will prove

Theorem 0.1 *Assume V satisfies* (V) *and it has exactly ℓ nondegenerate critical values $c_1 < \ldots < c_\ell$, and f satisfies* (f). *If*

$$\{T \max_{x \in \mathbf{R}^n} |\nabla V(x)| + T^{\frac{1}{2}} \| f \|_2\}^2 < 8\pi^2 \min_{2 \leq j \leq \ell}(c_j - c_{j-1}) \tag{0.2}$$

then (0.1) possesses at least ℓ T-periodic solutions.

In Theorem 0.1 $|\cdot|$ denotes the usual norm in $\mathbf{R}^n$ and $\|\cdot\|_2$ the usual norm in $L^2[0,T]$. Our hypothesis in V implies that $\ell \geq 2^n$. In [3] a multiplicity result is obtained when the functional associated to (0.1) is assumed to have only nondegenerate critical points. Under this condition (0.1) possesses at least 2^n T-periodic solutions. However this condition is hard to check in a concrete problem.

We will prove this theorem as a particular case of a general result for a Lagrangean system having T^n as its configuration space. In this context we mention a recent result of Chang, Long and Zehnder [2] on the n-pendulum, and also the previous results on the double pendulum by Fournier and Willem [6]. In [2] a general critical point result obtained by using relative category is applied to the n-pendulum equation yielding at least 2^n T-periodic solutions. Our results, when applied to the n-pendulum equation yields a similar result, and it is in a certain sense more general.

Our method is based on the notion of relative category introduced by Fadell in [4], and later and independently by Fournier and Willem in [6]. The idea is to relate the level sets of a finite dimensional functional with the level sets of the functional corresponding to the differential equation.

The computation of the relative category in concrete cases is difficult in general, and in order to get useful estimates of it we often need to use cohomological arguments. In this paper we keep our results in a simple form in such a way we do not need to use cohomological estimates. See Remarks 3.2 and 5.1.

The ideas we use to study Lagrangean systems can also be applied to some elliptic problems with Neumann boundary conditions. Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain and let $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u on the boundary $\partial\Omega$. We consider the Neumann problem

$$-\Delta u + u = p(u) + h(x) \quad x \in \Omega \tag{0.3}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega. \tag{0.4}$$

where p satisfies

(p1) $\quad p \in C^1(\mathbf{R}, \mathbf{R})$ is a bounded function,

and h satisfies

(h) $\quad h \in C^1(\Omega, \mathbf{R})$.

Let us define

$$\bar{h} = \frac{1}{\mid \Omega \mid} \int_\Omega f(x)\, dx \quad \text{and} \quad P(u) = \int_0^u p(s)\, ds \tag{0.5}$$

and let us consider

$$v(s) = s^2 - \bar{h} - P(s). \tag{0.6}$$

Let $|\cdot|_\infty$ denotes the L^∞ norm and $\mid \Omega \mid$ the Lebesgue measure of Ω. Then we prove the following theorem.

Theorem 1.5.1

Assume that h satisfies (h), *and p satisfies* (p1) *and the function v has ℓ critical values $c_1 < \ldots < c_\ell$ each of them being a strict local maximum or minimum. If*

$$\{\| h - \bar{h} \|_2 + | p |_\infty | \Omega |^{\frac{1}{2}}\}^2 < 4 \mid \Omega \mid \min_{2 \le j \le \ell} (c_j - c_{j-1}) \tag{0.7}$$

then (0.4) possesses at least ℓ classical solutions.

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1 Ljusternik-Schnirelmann Relative Category

In this section we present the notion of Ljusternik-Schnirelmann (L.S.) relative category as introduced by Fadell [4]. We give the basic definitions and properties and we applied it to prove a general critical point theorem that we use later in the applications. For the classical notion of (L.S.) category we refer the reader to [12].

Definition 1.1 *Let M be a topological space and let $U \subset M$. U is called* categorical in M *if it is contractible to a point in M.*

Definition 1.2 *Let (M, A) be a topological pair with $A \neq \emptyset$ and closed in M. Let $U \subset M$ such that $A \subset U$, then U is called* categorical relative to A in M *if there is a homotopy $H : [0,1] \times U \longrightarrow M$ such that*

$$H(0,x) = x \quad \forall x \in U$$
$$H(1,x) \in A \quad \forall x \in U$$
$$H(t,x) = x \quad \forall x \in A,\ \forall t \in [0,1].$$

Definition 1.3 *Let $A \subset X \subset M$. An open cover Ω of X is called* admissible *if Ω has the form $\Omega = \{W, U_1, U_2, \dots\}$ where $A \subset W$, W is categorical relative to A, and each U_i is categorical.*

Definition 1.4 *If $A \subset X \subset M$ with A nonempty and closed in M, then the* (L.S) relative category of (X, A) in M, *denoted by $\mathrm{cat}_M(X, A)$, is n if there exist an admissible cover $\Omega = \{W, U_1, \dots, U_n\}$, and n is minimal with this property. If such an n does not exist, then we say $\mathrm{cat}_M(X, A) = \infty$.*

Remark 1.1 There are other notions of relative category. What we defined here corresponds to the so called relative category star introduced in [4]. Fournier and Willem in [6] introduced a slightly different notion, see also [2].

The relative category possesses properties analogous to the classical category. Let us assume that M is normal, locally contractible and path connected, and $A \subset M$ is a closed set for which there exists an open set $V \supset A$ that is categorical relative to A. Then we have:

1. *Monotonicity:* If $A \subset X \subset Y \subset M$ then $\mathrm{cat}_M(X, A) \leq \mathrm{cat}_M(Y, A)$.

2. *Subadditivity:* If $A \subset X \subset M$ and $Y \subset M$ then

$$\mathrm{cat}_M(X \cup Y, A) \leq \mathrm{cat}_M(X, A) + \mathrm{cat}_M(Y)$$

where $\mathrm{cat}_M(Y)$ denotes the usual (L.S.) category.

3. *Invariance:* If $h : (M, A) \longrightarrow (M', A')$ is a homeomorphism of pairs then

$$\mathrm{cat}_M(X, A) = \mathrm{cat}_{M'}(h(X), A').$$

4. *Homotopy:* If $A \subset X \subset M$ and $h : [0,1] \times X \longrightarrow M$ is a homotopy relative to A such that $h(0, x) = x \ \forall x \in X$ then

$$\mathrm{cat}_M(X, A) \leq \mathrm{cat}_M(h(X, 1), A).$$

5. *Continuity:* If $A \subset X$, X closed in M, then there is a closed neighborhood U of X such that

$$\mathrm{cat}_M(X, A) = \mathrm{cat}_M(U, A).$$

The hypotheses on M needed for proving these properties are satisfied, for example, when M is a connected Riemannian manifold modeled on a Hilbert space. We refer to [4] for more details and the proof of these properties.

Remark 1.2 The computation of the relative category is often difficult. Estimates can be obtained by using the cohomology ring of the pairs involved. See [2], [4]. In our applications we consider cases where the computation of relative category is simple. More refined versions of our results may require the use of cohomology theory in order to estimate the relative category of certain pairs.

In what follows we use the relative category to prove a critical point theorem that we apply later to study some differential equations. We will need to introduce the Palais-Smale condition.

Definition 1.5 *Let M be a Riemannian manifold. A functional of class C^1, $I : M \to \mathbf{R}$ satisfies the* Palais-Smale condition (P.S.) *if for every sequence $\{x_n\}_{n\in\mathbf{N}} \subset M$ such that*

$\{I(x_n)\}_{n\in\mathbf{N}}$ *is bounded and* $\|\nabla I(x_n)\| \to 0$ *as* $n \to \infty$
there exists a convergent subsequence.

We consider the following notation when f is a functional on M and $a, b \in \mathbf{R}$:
$f^b = \{x \in M \ / \ f(x) \leq b\}$, $f_a^b = \{x \in M \ / \ a \leq f(x) \leq b\}$ and
$K_b = \{x \in M \ / \ \nabla f(x) = 0, \ f(x) = b\}$.

Theorem 1.1 (The Deformation Lemma) *Let M be a complete Riemannian manifold and $I : M \to \mathbf{R}$ a C^1 functional satisfying the Palais-Smale condition. For every $c \in \mathbf{R}$, $\bar{\epsilon} > 0$ and U a neighborhood of K_c, there exists $\epsilon > 0$, $\epsilon > \bar{\epsilon}$ and a homotopy $\eta : [0,1] \times M \to M$ such that*

D0. $\eta(0,x) = x \quad \forall x \in M$.

D1. $\eta(t,\cdot)$ *is a homeomorphism* $\forall t \in [0,1]$.

D2. $\eta(t,x) = x$ *if* $| I(x) - c | \geq \bar{\epsilon} \quad \forall t \in [0,1]$.

D3. $I(\eta(t,x)) \leq I(x) \quad \forall x \in M, \ \forall t \in [0,1]$.

D4. $\eta(1, I^{c+\epsilon} \setminus U) \subset I^{c-\epsilon}$.

In case $K_c = \emptyset$ or more generally, if for numbers a, b we have $K_c = \emptyset \ \forall c \in [a,b]$ then the homotopy η can be obtained so that it satisfies D0, D1, D3, *and also*

D2′. $\eta(t,x) = x$ *if* $I(x) - b \geq \bar{\epsilon}$ *or* $I(x) - a \leq -\bar{\epsilon} \ \forall t \in [0,1]$.

D4′. $\eta(1, I^b) \subset I^a$.

For the proof of the Deformation Lemma we refer to [8] and [9]. Now we can prove our critical point theorem.

Theorem 1.2 *Let M be a complete, connected Riemannian manifold and $I : M \to \mathbf{R}$ be a C^1 functional satisfying the (P.S.) condition. Let $a < b$ and $\sigma > 0$. If $A \subset I^{a-\sigma}$ and*

$$m = \mathrm{cat}_M(I^b, A) - \mathrm{cat}_M(I^a, A)$$

then I possesses at least m critical points in I_a^b.

Proof. For $j \geq 1$ we define the following class of subsets of M

$$\Sigma_j = \{X \ / \ A \subset X \subset I^b, X \text{ closed and } \mathrm{cat}_M(X, A) \geq j\}$$

and we consider the values

$$c_j = \inf_{X \in \Sigma_j} \sup_{x \in X} I(x).$$

Let $m_0 = \mathrm{cat}_M(I^a, A)$. If $j > m_0$ and $X \in \Sigma_j$ then $X \cap I_a^b \neq \emptyset$, so that

$$\sup_{x \in X} I(x) \geq a,$$

thus the numbers c_j are bounded from below by a. Because $X \subset I^b$, they are also bounded by above by b. Since the classes Σ_j satisfies $\Sigma_{j+1} \subset \Sigma_j$ and by the estimates we obtained above we have

$$a \leq c_{m_0+1} \leq \ldots \leq c_{m_0+m} \leq b.$$

Let us assume that for some $j \geq m_0$, $\ell \geq 1$ we have

$$c_{j+1} = \ldots = c_{j+\ell} \equiv c.$$

Then we show that $\mathrm{cat}_M(K_c) \geq \ell$. By the continuity property of the usual (L.S.) category (see [12]) there exists a neighborhood U of K_c such that

$$\mathrm{cat}_M(U) = \mathrm{cat}_M(K_c)$$

and since $A \subset I^{a-\sigma}$ we can choose U such that $U \cap A = \emptyset$. Let us consider $\bar{\epsilon} = \frac{1}{2}(c-a+\sigma) > 0$ and choose $\epsilon > 0$ and η as given by the Deformation Lemma. Choose $X \in \Sigma_{j+\ell}$ such that

$$\sup_{x \in X} I(x) \leq c + \epsilon. \tag{1.1}$$

Using the subadditivity of the relative category we have

$$\mathrm{cat}_M(X, A) \leq \mathrm{cat}_M(X \setminus U, A) + \mathrm{cat}_M(U). \tag{1.2}$$

If we assume $\mathrm{cat}_M(K_c) \leq \ell - 1$, then from (1.2) we get

$$\mathrm{cat}_M(X \setminus U, A) \geq \mathrm{cat}(X, A) - \ell + 1 \geq j + 1. \tag{1.3}$$

By (D4) in the Deformation Lemma, and the homotopy property of relative category we obtain that

$$\eta(1, X \setminus U) \in \Sigma_{j+1} \quad \text{and} \quad \sup_{x \in \eta(1, X \setminus U)} I(x) \leq c - \epsilon \tag{1.4}$$

contradicting the definition of c. Using similar arguments we can show that every value c_j is a critical value of I, if $m_0 + 1 \leq j \leq m_0 + m$. Thus I possesses at least m critical points.□

2 Lagrangean Systems in T^n

In this section we consider Lagrangean systems having the n-dimensional torus as configuration space. Here we introduce a mathematical framework to study the existence of periodic motions for such systems.

A Lagrangean system is characterized by its configuration space and its Lagrangean. We assume the Lagrangean has the following form:

$$L(q, p, t) = K(q, p, t) + G(q, p, t) - V(q) + F(q, t), \tag{2.1}$$

where $p, q \in \mathbf{R}^n$, $t \in \mathbf{R}$.

Remark 2.1 In mechanics the term K represents the kinetic energy, the term G is associated to gyroscopic forces, V is the potential energy and F represents the external forces.

Let $\tau_i > 0$, $1 \leq i \leq n$, $k \in \mathbf{Z}^n$, and denote $\tau k = (\tau_1 k_1, \ldots, \tau_n k_n)$. Also let $T > 0$ and define $\mathcal{L}_s(\mathbf{R}^n, \mathbf{R}^n)$ as the space of $n \times n$, symmetric matrices with coefficients in $\mathbf{R}$. We make the following general hypotheses on the terms involved in L:

(L1) We assume that $K(q,p,t) = \frac{1}{2}(Q(q,t)p,p)$ where Q is a function $Q : \mathbf{R}^n \times \mathbf{R} \to \mathcal{L}_s(\mathbf{R}^n,\mathbf{R}^n)$ of class C^1 and it satisfies:

i) $Q(q,t+T) = Q(q+\tau k,t) = Q(q,t) \quad \forall t \in \mathbf{R},\ q \in \mathbf{R}^n,\ k \in \mathbf{Z}^n$, and

ii) $\exists \lambda > 0$ such that $(Q(q,t)\xi,\xi) \geq \lambda \mid \xi \mid^2 \ \forall t \in \mathbf{R},\ q \in \mathbf{R}^n,\ \xi \in \mathbf{R}^n$.

(L2) We assume that $G(q,p,t) = (g(q,t),p)$ where $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is of class C^1 and it satisfies

$$g(q,t+T) = g(q+\tau k,t) = g(q,t) \quad \forall t \in \mathbf{R},\ q \in \mathbf{R}^n,\ k \in \mathbf{Z}^n.$$

(L3) $V : \mathbf{R}^n \to \mathbf{R}$ is of class C^1 and it satisfies

$$V(q+\tau k) = V(q) \quad \forall q \in \mathbf{R}^n,\ \forall k \in \mathbf{Z}^n.$$

(L4) We assume that $F(q,t) = (f(t),q)$ where $f : \mathbf{R} \to \mathbf{R}^n$ is continuous and it satisfies

i) $f(t+T) = f(t) \quad \forall t \in \mathbf{R}$, and

ii) $\int_0^T f(t)\,dt = 0$.

Here and in the rest of the paper $(\cdot,\cdot)$ denotes the usual inner product in $\mathbf{R}^n$ and $|\cdot|$ its corresponding norm.

Under the hypotheses (L1)-(L4) we can set up the mathematical framework to study T-periodic motions of the Lagrangean system L. Those periodic motions are T-periodic solutions of the Euler-Lagrange equation

$$\frac{d}{dt}L_p(q,\dot{q},t) - L_q(q,\dot{q},t) = 0. \tag{2.2}$$

We consider $W_T^{1,2}$ the usual Sobolev space of T-periodic functions under the inner product

$$< q,\xi > = \int_0^T (q,\xi) + (\dot{q},\dot{\xi})\,dt$$

for $q,\xi \in W_T^{1,2}$. The space $E \equiv W_T^{1,2}$ is a Hilbert space and we denote the induced norm by $\|\cdot\|_E$. We will denote by L_T^2 the space of square integrable functions on $[0,T]$ with values in $\mathbf{R}^n$. We denote its usual norm by $\|\cdot\|_2$.

In E we define the following functional:

$$I(q) = \int_0^T L(q,\dot{q},t)\,dt.$$

This functional is well defined in E and it is of class C^1. Moreover critical points of I are the classical T-periodic solutions of (2.2). To prove this we can use the arguments given in [10]. Thus the problem of finding T-periodic solutions for (2.2) is reduced to that of finding critical points of I in E.

In E we introduce the following action of $\mathbf{Z}^n$. For $q \in E$ and $k \in \mathbf{Z}^n$ we define

$$kq = q + \tau k = (q_1 + \tau_1 k_1, \ldots, q_n + \tau_n k_n).$$

By analyzing the Fourier series of the elements of E one can see that

$$E/\mathbf{Z}^n \cong \tilde{E} \times \mathrm{T}^n \equiv M$$

where $\tilde{E} = \{q \in E \ / \ \int_0^T q(t)\, dt = 0\}$, and T^n is the n-dimensional torus.

The periodicity in q assumed in (L1)-(L3), and (L4) (ii) shows that I is invariant under the $\mathbf{Z}^n$-action defined above, i.e.,

$$I(q + \tau k) = I(q) \quad \forall q \in E, \forall k \in \mathbf{Z}^n.$$

Thus we can define I on the quotient space M and study the critical points of I in M. We note that M is a complete, connected Riemannian manifold.

Proposition 2.1 *If the Lagrangean defined in (2.1) satisfies* (L1)-(L4) *then the functional I satisfies the (P.S.) condition in M.*

Proof. When the functional I does not contain the term G, this proposition has been proved in [2] and [11]. A slight modification of either of these gives our result. We omit the details.□

The following result has been proved in [1], [5] and [11], and it reflects the fact that the (L.S.) category of T^n is $n + 1$.

Theorem 2.1 *Assume the Lagrangean L defined in* (2.1) *satisfies* (L1)-(L4). *Then I possesses at least $n + 1$ critical points.*

3 Multiple critical points inherited by the potential energy V

Since the potential energy function V satisfies (L3), we can view it as a function on T^n. Then by using (L.S.) category one can prove that V has at least $n + 1$ critical points in T^n. In Theorem 2.1 we found that that I also has at least $n + 1$ critical points. However the function V on T^n can have more critical points. For example if V is a C^2 Morse function, then V has at least 2^n critical points. In this section we will prove some results that show that I on M has at least the same number of critical points as V on T^n when some extra assumptions are imposed on L.

We start with a preliminary result. In what follows we use the following notation

$$\nu = \sup_{x \in \mathrm{T}^n} | \nabla V(x) |$$

and

$$\gamma = \sup_{x \in \mathrm{T}^n,\, t \in \mathbf{R}} | g(x,t) |,$$

and we let $\bar{V}(q)$ be the negative of the potential energy, i.e. $\bar{V}(q) = -v(q)$.

Theorem 3.1 *Let $a < b < c$ be real numbers and assume a is a regular value of $\bar{V}$. Let*

$$m = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^a) - \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, \bar{V}^a).$$

If

$$\{2\pi\gamma + T\nu + T^{\frac{1}{2}} \parallel f \parallel_2\}^2 < 8\pi^2\lambda(b-a) \tag{3.1}$$

then I possesses at least m critical points with critical values in $[Ta, Tc]$.

Before proving this theorem we give the following proposition based on the Deformation Lemma.

Proposition 3.1 *Let M^n be a compact manifold and let $f : M^n \longrightarrow \mathbf{R}$ be a C^1 functional . Assume $a < b < c$, and f does not have critical points in f_a^b. Then*

$$\mathrm{cat}_{M^n}(f^c, f^a) = \mathrm{cat}_{M^n}(f^c, f^b) \tag{3.2}$$

Proof. For $\alpha \in \mathbf{R}$, a regular value of f, f^α is a submanifold with boundary in M^n, then there exist an open set U, and a homotopy $h_1 : [0,1] \times U \longrightarrow M^n$ such that

$$\begin{array}{rcl} h_1(0,x) & = & x \quad \forall x \in U \\ h_1(1,x) & \in & f^\alpha \quad \forall x \in U \\ h_1(t,x) & = & x \quad \forall x \in f^\alpha,\ \forall t \in [0,1], \end{array} \tag{3.3}$$

in other words f^α has a neighborhood U, that is categorical relative to f^α. In order to prove (3.2) we first show that

$$\mathrm{cat}_{M^n}(f^c, f^a) \leq \mathrm{cat}_{M^n}(f^c, f^b). \tag{3.4}$$

Assume $\mathrm{cat}_{M^n}(f^c, f^a) = m$. Then there is an admissible cover of f^c relative to f^a, $\Omega = \{U_1, \ldots, U_m, W\}$. Let W' be a neighborhood of f^b categorical relative to f^b. Since b is a regular value, such a W' exits. Then $\Omega' = \{U_1, \ldots, U_m, W'\}$ is an admissible cover of f^c relative to f^b. This proves (3.4).

Now let us show that

$$\mathrm{cat}_{M^n}(f^c, f^a) \geq \mathrm{cat}_{M^n}(f^c, f^b). \tag{3.5}$$

Let $\mathrm{cat}_{M^n}(f^c, f^b) = m$. Let $\Omega = \{U_1, \ldots, U_m, W\}$ be an admissible covering of f^c relative to f^b. We claim that Ω is also an admissible cover of f^c relative to f^a. This will complete the proof of (3.5).

Let h_3 be a homotopy $h_3 : [0,1] \times W \longrightarrow M^n$ such that

$$\begin{array}{rcl} h_3(0,x) & = & x \quad \forall x \in W \\ h_3(1,x) & \in & f^b \quad \forall x \in W \\ h_3(t,x) & = & x \quad \forall x \in f^b,\ \forall t \in [0,1]. \end{array} \tag{3.6}$$

Let U be a neighborhood of f^a and h_1 a homotopy as in (3.3). Since M^n is compact there is $a' > a$ such that $f^{a'} \subset U$. Using (D4') of the Deformation Lemma with $\bar{\epsilon} < a' - a$, in the interval $[a', b]$ we find a homotopy $h_2 : [0,1] \times M^n \longrightarrow M^n$ such that

$$\begin{array}{rcl} h_2(0,x) & = & x \quad \forall x \in M^n \\ h_2(1,x) & \in & f^{a'} \quad \forall x \in V^b \\ h_2(t,x) & = & x \quad \forall x \in f^a,\ \forall t \in [0,1]. \end{array} \tag{3.7}$$

Now we define a homotopy $H : [0,1] \times W \longrightarrow M^n$ by

$$H(t,x) = \begin{cases} h_3(3t,x) & 0 \leq t \leq 1/3 \\ h_2(3t-1, h_3(1,x)) & 1/3 \leq t \leq 2/3 \\ h_1(3t-2, h_2(1, h_3(1,x))) & 2/3 \leq t \leq 1. \end{cases} \tag{3.8}$$

The function H is continuous and it is easy to check that it makes W categorical relative to f^a.□

Proof of Theorem 3.1. Let σ small enough so that $\bar{V}$ does not have critical values in $[a-\sigma, a]$, and let us define $A = \bar{V}^{a-\sigma}$. Using Proposition 3.1 we see that

$$\mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, \bar{V}^a) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, A) \quad \text{and} \tag{3.9}$$

$$\mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^a) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, A). \tag{3.10}$$

If we can show that

$$\mathrm{cat}_M(I^{Tc}, A) - \mathrm{cat}_M(I^{Ta}, A) \geq m, \tag{3.11}$$

then an application of Theorem 1.2 completes the proof.

It is easy to see that $\bar{V}^c \subset I^{Tc}$. Then by the monotonicity property of relative category and (3.10) we obtain that

$$\mathrm{cat}_M(I^{Tc}, A) \geq \mathrm{cat}_M(\bar{V}^c, A) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, A) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^a). \tag{3.12}$$

Here we used that $\tilde{E}$ is contractible to a point.

Let us show now that

$$\mathrm{cat}_M(I^{Ta}, A) \leq \mathrm{cat}_{T^N}(\bar{V}^b, \bar{V}^a). \tag{3.13}$$

For $q \in E$ we write $\bar{q} = \frac{1}{T}\int_0^T q(t)\,dt$, and $\tilde{q} = q - \bar{q}$. We note that $\tilde{q} \in \tilde{E}$. Let $q \in I^{Ta}$ then

$$\int_0^T \frac{1}{2}(Q(q,t)\dot{q},\dot{q}) + (g(q,t),\dot{q}) + \bar{V}(q) + (f(t),q)\,dt \leq Ta. \tag{3.14}$$

Using (L1), (L2), (L4), the Schwarz inequality and the definition of γ, from (3.14) we have

$$\int_0^T \bar{V}(q)\,dt \leq Ta - \frac{\lambda}{2} \parallel \dot{q} \parallel_2^2 + \gamma T^{\frac{1}{2}} \parallel \dot{q} \parallel_2 + \parallel f \parallel_2 \parallel \tilde{q} \parallel_2 . \tag{3.15}$$

By the Mean Value Theorem, Schwarz inequality and definition of ν we have

$$|\int_0^T \bar{V}(\tilde{q}+\bar{q}) - \bar{V}(\bar{q})\,dt\,| \leq \int_0^T \nu \mid \tilde{q} \mid dt \leq \nu T^{\frac{1}{2}} \parallel \tilde{q} \parallel_2 . \tag{3.16}$$

From (3.15) and (3.16) we see that

$$T\bar{V}(\bar{q}) \leq Ta - \frac{\lambda}{2} \parallel \dot{q} \parallel_2^2 + \gamma T^{\frac{1}{2}} \parallel \dot{q} \parallel_2 + (\parallel f \parallel_2 + \nu T^{\frac{1}{2}}) \parallel \tilde{q} \parallel_2 \tag{3.17}$$

and then, using Wirtinger's inequality in (3.17) we obtain that

$$T\bar{V}(\bar{q}) \leq Ta - \frac{\lambda}{2} \parallel \dot{q} \parallel_2^2 + \{\gamma T^{\frac{1}{2}} + \frac{T}{2\pi}(\parallel f \parallel_2 + \nu T^{\frac{1}{2}}\} \parallel \dot{q} \parallel_2 \tag{3.18}$$

By maximizing the quadratic $p(x) = -\alpha x^2 + \beta x$, from (3.18) we find that

$$T\bar{V}(\bar{q}) \leq Ta + \frac{1}{2\lambda}\{\gamma T^{\frac{1}{2}} + \frac{T}{2\pi}(\parallel f \parallel_2 + \nu T^{\frac{1}{2}})\}^2. \tag{3.19}$$

Then by our condition (3.1) we obtain that $\bar{V}(\bar{q}) < b$. We have shown that if $\tilde{q} + \bar{q} \in I^{Ta}$, then $\bar{q} \in \bar{V}^b$.

Let us define the following homotopy:

$$\begin{array}{rcl} H : [0,1] \times I^{Ta} & \longrightarrow & M \\ (t,q) & \longmapsto & (1-t)\tilde{q} + \bar{q}. \end{array} \tag{3.20}$$

Clearly H is continuous and it satisfies:

$$\begin{array}{rcll} H(0,q) & = & q & \forall q \in I^{Ta} \\ H(1,q) & \in & \bar{V}^b & \forall q \in I^{Ta} \quad \text{and} \\ H(t,q) & = & q & \forall q \in \bar{V}^b. \end{array} \tag{3.21}$$

Consequently, by the homotopy property of relative category and (3.9) we have

$$\mathrm{cat}_M(I^{Ta}, A) \leq \mathrm{cat}_M(\bar{V}^b, A) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, A) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, \bar{V}^a). \tag{3.22}$$

Here again we have used that $\tilde{E}$ is contractible. Thus we have proved (3.13). From (3.12) and (3.13) we conclude that (3.11) holds, so finishing the proof.□

Remark 3.1 We note that by using Theorem 3.1 for the function $\bar{V}$ in T^n we can conclude that given

$$m = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^a) - \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^b, \bar{V}^a)$$

$\bar{V}$ has at least m critical points in $\bar{V}_b^c$. $\bar{V}$ may have some critical points in $\bar{V}_a^b$ that the functional I will 'lose'.

In the following theorem we make a global assumption on the potential energy V and its critical points in order to obtain a multiplicity result for the critical points of I.

Definition 3.1 *An isolated critical value c of $\bar{V}$ is said to be* nontrivial *if for every $\epsilon > 0$*

$$\mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^{c-\epsilon}) > 0. \tag{3.23}$$

Theorem 3.2 *Let us assume that $\bar{V}$ has only isolated nontrivial critical values. Let $c_1 < c_2 < \ldots < c_\ell$ be those values, and let us write*

$$C(\bar{V}) = \min_{1 \leq j \leq \ell - 1}(c_{j+1} - c_j). \tag{3.24}$$

If

$$\{2\pi\gamma + \nu T + T^{1/2} \parallel f \parallel\}^2 < 8\pi^2 \lambda C(\bar{V}), \tag{3.25}$$

then I possesses at least ℓ critical points.

Proof. Since I is bounded from below and it satisfies the (P.S.) condition, we have that $\inf_M I = Tc_0$ is a critical value of I, and clearly $c_0 \leq c_1$. We show next, by using Theorem 1.2, that for every j, $2 \leq j \leq \ell$, I possesses at least one critical point $q_j \in I^{Tc_j}_{Tc_{j-1}}$. Let $c = c_j$, $b = c_j - \epsilon$ and $a = c_{j-1} + \epsilon$ with $\epsilon > 0$ to be determined later. Since c_j and c_{j-1} are isolated critical values of $\bar{V}$, by using Proposition 3.1 we have

$$\mathrm{cat}_{\mathrm{T}^n}(\bar{V}^{c_j}, \bar{V}^{c_j - \epsilon}) = \mathrm{cat}_{\mathrm{T}^n}(\bar{V}^c, \bar{V}^a) > 0 \tag{3.26}$$

and

$$\mathrm{cat}(\bar{V}^b, \bar{V}^a) = 0. \tag{3.27}$$

Since $C(\bar{V}) \leq (c_j - c_{j-1})$, from (3.25) we see that there is $\epsilon > 0$ so that

$$\{2\pi\gamma + \nu T + T^{1/2} \parallel f \parallel_2\}^2 < 8\pi^2 \lambda (b - a). \tag{3.28}$$

By (3.26)-(3.28) we see that the hypotheses of Theorem 3.1 are satisfied, so that we can conclude that I possesses at least one critical point in $I^{Tc}_{Ta} = I^{Tc_j}_{Tc_{j-1}+\epsilon}$. Since we can do this for every $j = 2, \ldots, \ell$ the proof is complete.□

Remark 3.2 We could obtain other results in the spirit of Theorem 3.2 by combining Theorem 3.1 with adequate assumptions on the structure of critical points of $\bar{V}$. Here we choose, in some way, the simplest situation.

Remark 3.3 If we assume V is a C^2 Morse function, then, after assuming condition (3.25), we can prove that I has as many critical points as V. The fact that every critical value of $\bar{V}$ is nontrivial is seen by using the 'cell-attaching' argument. See [8].

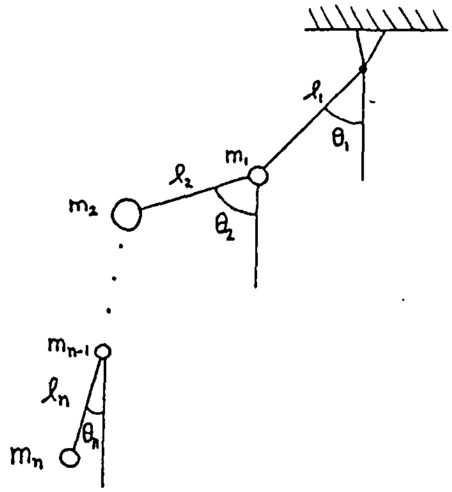

Figure 1: The n-pendulum

4 Application to the n-pendulum

In Section 3 we proved a general result for the existence of multiple periodic motions for a general Lagrangean system in T^n. Here we apply those results to a specific example: the n-pendulum.

We will consider two cases. In the first case we study the n-pendulum with fixed support and with an external force acting on it. In the second case we consider an n-pendulum with moving support, but without external forces.

Let us start by describing the idealized physical situation. We have a system composed of n particles, each of them of mass $m_i > 0$, $i = 1, \ldots, n$, joined by weightless rods of length $\ell_i > 0$, $i = 1, \ldots, n$. See Figure 1. Let (x_i, y_i) represents the position of the particle i in the coordinate system given in Figure 1. The potential energy given by the action of gravity forces has the form

$$V = -g \sum_{i=1}^{n} m_i y_i \tag{4.1}$$

and the kinetic energy is given by

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2). \tag{4.2}$$

Introducing the angular variable $\theta = (\theta_1, \ldots, \theta_n)$, from (4.1) and (4.2) we obtain

$$V(\theta) = -g \sum_{i=1}^{n} M_i \ell_i \cos\theta_i \tag{4.3}$$

and

$$K(\theta, \dot{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} Q_{ij}(\theta) \dot{\theta}_i \dot{\theta}_j, \tag{4.4}$$

where

$$M_i = \sum_{j=i}^{n} m_j \tag{4.5}$$

and

$$Q_{ij}(\theta) = \begin{cases} M_i \ell_i^2 & i = j \\ M_i \ell_i \ell_j \cos(\theta_i - \theta_j) & i > j \end{cases} \tag{4.6}$$

with $Q_{ij}(\theta) = Q_{ji}(\theta)$. If we define $Q(\theta)$ to be the matrix with entries $Q_{ij}(\theta)$ then the kinetic energy can be written as

$$K(\theta, \dot{\theta}) = \frac{1}{2}(Q(\theta)\dot{\theta}, \dot{\theta}). \tag{4.7}$$

A straightforward computation gives the following proposition.

Proposition 4.1 *The matrix $Q(\theta)$ is positive definite and there is $\lambda > 0$ such that*

$$(Q(\theta)\xi, \xi) \geq \lambda \mid \xi \mid^2 \quad \forall \theta \in \mathbf{R}^n,\ \forall \xi \in \mathbf{R}^n. \tag{4.8}$$

Remark 4.1 The constant λ only depends on the mass of the particles $\{m_i\}_{i=1}^n$ and the length of the rods $\{\ell_i\}_{i=1}^n$. Actually one can prove that

$$\lambda = c(n)\ \min\{m_i,\ 1 \leq i \leq n\}\ \min\{l_i,\ 1 \leq i \leq n\},$$

where $c(n)$ is a constant depending only on the number of particles n.

We have an external force acting on the system that is represented by a continuous T-periodic function $f : \mathbf{R}^n \to \mathbf{R}^n$ and we assume it satisfies

$$\int_0^T f(t)\, dt = 0. \tag{4.9}$$

Then we can define the Lagrangean associated to the system:

$$L(\theta, \dot{\theta}, t) = \frac{1}{2}(Q(\theta)\dot{\theta}, \dot{\theta}) - V(\theta) + (f(t), \theta). \tag{4.10}$$

We refer the reader to the book by Sommerfeld [13] for more details about the derivation of the Lagrangean. From the definition of Q and V, Proposition 4.1 and condition (4.9) on f we easily see that the Lagrangean L, defined in (4.10) satisfies (L1)-(L4).

Now we introduce a condition on V that determines the nature of its critical points:

(N) $\quad V$ has exactly 2^n different critical values.

This condition is really a condition on the mass of the particles and the length of the rods. If $\alpha \in \{0, 1\}^n$ then we define

$$v(\alpha) = g \sum_{i=1}^n (-1)^{\alpha_i} M_i \ell_i. \tag{4.11}$$

Then we can rewrite condition (N) as

$$v(\alpha) \neq v(\alpha') \quad \text{if} \quad \alpha \neq \alpha' \quad \alpha, \alpha' \in \{0, 1\}^n. \tag{4.12}$$

Condition (N) then induces a linear order on the finite set $\{0, 1\}^n$ defined by

$$\alpha \prec \beta \quad \text{if and only if} \quad v(\alpha) < v(\beta). \tag{4.13}$$

Remark 4.2 One can show that condition (N) is satisfied generically on the mass of the particles $\{m_i\}_{i=1}^n$ and the legth of the rods $\{l_i\}_{i=1}^n$. In fact, if $\alpha \neq \beta$ are in $\{0,1\}^n$ and

$$\Gamma_{\alpha,\beta} = \{(m,\ell) \in \mathbf{R}_+^{2n} \ / \ \sum_{i=1}^n (-1)^{\alpha_i} (\sum_{j=i}^n m_j)\ell_i \neq \sum_{i=1}^n (-1)^{\beta_i} (\sum_{j=i}^n m_j)\ell_i \ \}$$

where $m = (m_1, \ldots, m_n)$ and $\ell = (\ell_1, \ldots, \ell_n)$, then $\Gamma_{\alpha,\beta}$ is open and dense in $\mathbf{R}_+^{2n}$. Condition (N) is satisfied when (m, ℓ) belongs to

$$\bigcap_{\alpha \neq \beta \in \{0,1\}^n} \Gamma_{\alpha,\beta}$$

that is also dense and open in $\mathbf{R}_+^{2n}$.

Let us define now

$$C(V) = \min\{v(s(\alpha)) - v(\alpha) \ / \ \alpha \in \{0,1\}^n \setminus \mathbf{1}\} \tag{4.14}$$

where $s(\alpha)$ denotes the successor of α under $\prec$ and $\mathbf{1} = (1, \ldots, 1)$. Given the form of V we can easily find that

$$\nu = \max_{x \in \mathbf{R}^n} \mid \nabla V(x) \mid = g(\sum_{i=1}^n (M_i \ell_i)^2)^{\frac{1}{2}}. \tag{4.15}$$

Finally we note that under condition (N) the function N is a Morse function so that for every $\alpha \in \{0,1\}^n$, $v(\alpha)$ is a nontrivial critical value. Actually

$$\mathrm{cat}_{\mathrm{T}^n}(V^{v(\alpha)}, V^{v(\alpha)-\epsilon}) = 1. \tag{4.16}$$

The following theorem is a direct application of Theorem 3.2.

Theorem 4.1 *If the potential energy V satisfies condition* (N) *and if*

$$\{gT(\sum_{i=1}^n (M_i \ell_i)^2)^{\frac{1}{2}} + T^{\frac{1}{2}} \parallel f \parallel_2\}^2 < 8\pi^2 \lambda C(V) \tag{4.17}$$

then the forced n-pendulum equation possesses at least 2^n T-periodic solutions.

Remark 4.3 Our theorem is more general than Theorem 4.1 in [2] in the sense that we do not assume any specific order in the set $\{0,1\}^n$. However, as a trade-off, our condition (4.17) is more restrictive than the analogous condition in [2]. Assuming the order in the critical values considered in [2] and using Theorem 3.1, we can obtain their result. Figure 2 illuminates the situation in the case $n = 2$.

Remark 4.4 Condition (4.17) can not be satisfied if T is too large. It would be interesting to have a result in the spirit of Theorem 4.1 when T is large, or at least in certain ranges of T for which

$$T(\sum_{i=1}^n (M_i \ell_i)^2)^{\frac{1}{2}} > 4\pi \lambda C(V). \tag{4.18}$$

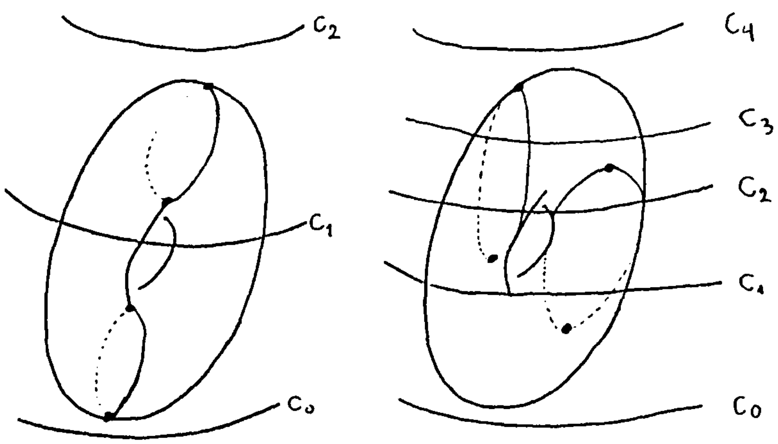

Figure 2: Comparing the two situations

Now we briefly discuss the case of an n-pendulum with moving support and without external forces. We assume that the support of the n-pendulum is moving vertically at a velocity given by a continuous T-periodic function $e : \mathbf{R} \to \mathbf{R}$. In this situation the kinetic energy of the system contains an additional term given by

$$G(\theta, \dot{\theta}, t) = -e(t) \sum_{i=1}^{n} M_i l_i \sin(\theta_i)\dot{\theta}_i. \tag{4.19}$$

Remark 4.5 To obtain (4.19) we modify in [13], page 195, the vertical velocity by adding the function $e(t)$ and proceed as in the other case. Actually the kinetic energy has an additional term that we neglect because it only depends on t.

Defining $g_i(\theta, t) = -e(t) M_i \ell_i \sin(\theta_i)$ we see that the function G satisfies our hypothesis (L2). Then we have the following theorem.

Theorem 4.2 *If V satisfies hypothesis* (N) *and if*

$$\sum_{i=1}^{n} (M_i \ell_i)^2 \{gT + \| e \|_2\}^2 < 8\pi^2 \lambda C(V) \tag{4.20}$$

then the equation of the n-pendulum with moving support possesses at least 2^n T-periodic solutions.

Proof. The proof is obtained in the same way Theorem 3.1 was obtained, only by noting that the specific knowledge of the function G leads to a better estimate.

By using the Schwarz inequality, first in $\mathbf{R}^n$ and then in L_T^2 we have

$$\int_0^T (g(\theta, t), \dot{\theta})\, dt \;\le\; \int_0^T | e | \Big(\sum_{i=1}^{n} (M_i \ell_i \sin\theta_i)^2\Big)^{\frac{1}{2}} | \dot{\theta} | \, dt \tag{4.21}$$

$$\le \Big(\sum_{i=1}^{n} (M_i \ell_i)^2\Big)^{\frac{1}{2}} \| e \|_2 \| \dot{\theta} \|_2 . \square \tag{4.22}$$

5 Application to the Neumann Problem

The ideas presented in Section 3 for studying the n-pendulum equation can also be applied to study the existence of solutions to certain elliptic problems with Neumann boundary condition.

We consider the problem

$$-\Delta u + u = p(u) + h(x) \quad x \in \Omega \tag{5.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega \tag{5.2}$$

where Ω denotes a smooth bounded domain in $\mathbf{R}^n$ and $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u on the boundary. Let $E \equiv H^1(\Omega)$ denotes the usual Sobolev space of functions in Ω with square integrable first derivatives.

We will assume that

(p1) $p : \mathbf{R} \to \mathbf{R}$ is a C^1 function, bounded in L^∞.

and

(h) $h : \bar{\Omega} \to \mathbf{R}$ is a C^1 function.

It is well known that the classical solutions of (5.1) corresponds to the critical points of the functional

$$I(u) = \int_\Omega (\frac{1}{2} \mid \nabla u \mid^2 + u^2 - P(u) - hu)\, dx \tag{5.3}$$

in E, where $P(u) = \int_0^u p(s)\, ds$. It can also be proved that the critical points of I, i.e. the weak solutions of (5.1) corresponds to classical solutions of (5.1). See for example [7].

Given the form of the functional I in which the quadratic term dominates for large u, it is easy to prove the following proposition.

Proposition 5.1 *The functional I satisfies the (P.S.) condition in E.*

Given h satisfying (h), let us define

$$\bar{h} = \frac{1}{\mid \Omega \mid} \int_\Omega h(x)\, dx \quad \text{and} \quad \tilde{h} = h - \bar{h}. \tag{5.4}$$

In Proposition 5.1 and in what follows $\mid \Omega \mid$ is the Lebesgue measure of Ω. Let us consider the real function

$$v(s) = s^2 - \bar{h}s - P(s) \tag{5.5}$$

and let us assume that

(p2) The function $v(s)$ possesses exactly ℓ distinct critical values, each of them being a strict local maximum or minimum.

Let $c_1 < c_2 < \ldots < c_\ell$ be the critical values of v and define

$$C(v) = \min_{2 \le j \le \ell} (c_j - c_{j-1}) \tag{5.6}$$

and let

$$| p |_\infty = \max_{s \in \mathbf{R}} | p(s) | . \tag{5.7}$$

The space E can be decomposed into $E \cong \tilde{E} \times \mathbf{R}$ where

$$\tilde{E} = \{ u \in E \;/\; \int_\Omega u \, dx = 0 \} \tag{5.8}$$

The following result, known as Poincaré-Wirtinger's inequality, assures the existence of a constant $\delta > 0$ so that for every $u \in \tilde{E}$

$$2\delta \, \| \tilde{u} \|_2^2 \le \| \nabla \tilde{u} \|_2^2 . \tag{5.9}$$

The proof of this inequality is easily obtained by using the spectral decomposition of the Laplacian under the given boundary conditions. Now we can prove the following theorem.

Theorem 5.1 *Assume p and h satisfies* (p1), (p2) *and* (h). *If*

$$\{ \| \tilde{h} \|_2 + | p |_\infty \, | \Omega |^{\frac{1}{2}} \}^2 \; < \; 4(\delta + 1) \, | \Omega | \, C(v) \tag{5.10}$$

then the Neumann problem (5.1) *possesses at least ℓ classical solutions.*

Proof. Since the functional I is bounded from below, and it satisfies the (P.S.) condition, we have that $\inf_E I = | \Omega | \, c_0$ is a critical value of I, and we can see that $c_0 \le c_1$.

Next we show that for every $j = 2, \ldots, \ell$ the functional I possesses at least one critical point $u_j \in I_{|\Omega| c_{j-1}}^{|\Omega| c_j}$. This is done via Theorem 1.2. Let us define $b = c_j$ and $a = c_{j-1} + 2\epsilon$, where $\epsilon > 0$ will be determined later. Let us consider $A = v^{c_{j-1}+\epsilon}$. We will complete the proof if we can prove that

$$\mathrm{cat}_E(I^{b|\Omega|}, A) \ge 1 \tag{5.11}$$

and

$$\mathrm{cat}_E(I^{a|\Omega|}, A) = 0. \tag{5.12}$$

Let $s \in v^b$ then we see that

$$\begin{aligned} I(s) &= \int_\Omega (s^2 - P(s) - hs) \, dx \\ &= | \Omega | \, (s^2 - \bar{h}s - P(s)) \\ &\le | \Omega | \, b, \end{aligned} \tag{5.13}$$

consequently $v^b \subset I^{b|\Omega|}$, and then by monotonicity property of relative category and the fact that $\tilde{E}$ is contractible

$$\begin{aligned} \mathrm{cat}_E(I^{b|\Omega|}, A) &\geq \mathrm{cat}_E(v^b, A) \\ &= \mathrm{cat}_{\mathbb{R}}(v^b, A). \end{aligned} \tag{5.14}$$

Since $b = c_j$ is a local maximum or minimum, and $A = v^{c_{j-1}+\epsilon}$, it is clear that

$$\mathrm{cat}_{\mathbb{R}}(v^b, A) = 1. \tag{5.15}$$

Then (5.14) and (5.15) give (5.11).

Let us show now that (5.12) holds. Let $u \in I^{a|\Omega|}$ and decompose it as $u = \tilde{u} + \bar{u}$ with $\tilde{u} \in \tilde{E}$. Then

$$\int_\Omega \frac{1}{2} \mid \nabla u \mid^2 + u^2 - P(u) - hu \, dx \leq a \mid \Omega \mid . \tag{5.16}$$

Using the Schwarz inequality, we have

$$\int_\Omega \bar{u}^2 - P(u) - \bar{h}\bar{u} \, dx \leq a \mid \Omega \mid - \int_\Omega \frac{1}{2} \mid \nabla \tilde{u} \mid^2 + \tilde{u}^2 \, dx + \| \tilde{h} \|_2 \| \tilde{u} \|_2 . \tag{5.17}$$

By the Mean Value Theorem and Schwarz inequality we obtain that

$$\begin{aligned} \int_\Omega P(u) - P(\bar{u}) \, dx &\leq \mid p \mid_\infty \int_\Omega \mid \tilde{u} \mid \, dx \\ &\leq \mid p \mid_\infty \mid \Omega \mid^{\frac{1}{2}} \| \tilde{u} \|_2 \end{aligned} \tag{5.18}$$

Then, by (5.17) and (5.18) we have

$$\begin{aligned} \mid \Omega \mid (\bar{u}^2 - P(\bar{u}) - \bar{h}\bar{u}) &\leq a \mid \Omega \mid - \int_\Omega \frac{1}{2} \mid \nabla \tilde{u} \mid^2 + \tilde{u}^2 \, dx \\ &+ (\| \tilde{h} \|_2 + \mid p \mid_\infty \mid \Omega \mid^{\frac{1}{2}}) \| \tilde{u} \|_2 . \end{aligned} \tag{5.19}$$

Using now the Poincaré-Wirtinger's inequality in (5.19) we see that

$$\mid \Omega \mid v(\bar{u}) \leq a \mid \Omega \mid - (\delta + 1) \| \tilde{u} \|_2^2 + (\| \tilde{h} \|_2 + \mid p \mid_\infty \mid \Omega \mid^{\frac{1}{2}}) \| \tilde{u} \|_2 . \tag{5.20}$$

By maximizing the quadratic $p(x) = -\alpha x^2 + \beta x$, from (5.20) we see that

$$\mid \Omega \mid v(\bar{u}) \leq a \mid \Omega \mid + \frac{(\| \tilde{h} \|_2 + \mid p \mid_\infty \mid \Omega \mid^{\frac{1}{2}})^2}{4(\delta + 1)}. \tag{5.21}$$

From (5.10), we can find an $\epsilon > 0$ so that, from (5.21) we obtain

$$v(\bar{u}) < b - \epsilon. \tag{5.22}$$

We have proved that if $u = \tilde{u}+\bar{u} \in I^{a|\Omega|}$ then $v(\bar{u}) < b-\epsilon$. We define now a homotopy:

$$\begin{array}{rcl} H : [0,1] \times I^{a|\Omega|} & \longrightarrow & E \\ (t,u) & \longmapsto & (1-t)\tilde{u} + \bar{u}. \end{array} \tag{5.23}$$

Clearly H is continuous and it satisfies

$H(0,u) = u \quad \forall u \in I^{a|\Omega|}$

$H(1,u) \in v^{b-\epsilon} \;\; \forall u \in I^{a|\Omega|}$

$H(t,u) = u \quad \forall t \in [0,1] \, \forall u \in A.$

By the homotopy property of relative category we have then that

$$\mathrm{cat}_E(I^{a|\Omega|}, A) \leq \mathrm{cat}_E(v^{b-\epsilon}, A) = \mathrm{cat}_{\mathbb{R}}(v^{b-\epsilon}, A), \tag{5.24}$$

and since $b - \epsilon = c_j - \epsilon$ and $a = c_{j-1} + \epsilon$ are regular values and there is no critical point in between by Proposition 3.1 we have

$$\mathrm{cat}_{\mathbb{R}}(v^{b-\epsilon}, A) = 0. \tag{5.25}$$

We conclude from (5.24) and (5.25) that (5.12) holds.□

Remark 5.1 In this context we could prove a theorem analogous to Theorem 3.1. Theorem 5.1 represents an extreme case where I possesses the same number of critical values as v. We could consider intermediate situations in which we would be able to relax condition (5.10) somewhat. The same results could be obtained for more general elliptic problems.

Remark 5.2 If we assume that p is a periodic function and that $\bar{h} = 0$, then a similar result can be obtained for the problem

$$-\Delta u = p(u) + f(x) \tag{5.26}$$

$$\frac{\partial u}{\partial \nu} = 0 \tag{5.27}$$

Here we would consider the corresponding critical point problem in the manifold $M = \tilde{E} \times S^1$. See also [11] for a related result.

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